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SUBGROUP IDEALS IN GROUPRINGS I

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1. Introduction

Let G be a group, ZG its integral groupring and A_G the augmentation ideal of ZG. Denote by $Q_n(G)=A_G^n/A_G^{n+1}$ and by G_i the ith term of the lower central series of G. Several authors have studied the structure of $Q_n(G)$ ([2], [4], [5]). It is well known that $Q_1(G)\simeq G_1/G_2$. Losey [2] proved that $Q_2(G)\simeq\{G_2/G_3\}\oplus\mathrm{Sp}^2(G_1/G_2)$ for any finitely generated group; where Sp^2 denotes the second symmetric product of G_1/G_2. Tahara has found the structure of $Q_3(G)$ for finite groups [4].

We are interested in the abelian group structure of the quotients $A_G^mA_H/A_G^{m+1}A_H$ where m is a positive integer. The case where $m=1$ is discussed in the author's earlier paper [1]. Here the author attempts to find the structure of $A_G^2A_H/A_G^3A_H$ where G is a finite split extension of a normal subgroup H by a subgroup K.

2. Notation and preliminaries

We will restrict ourselves to the notation of Losey and Tahara.

Let M be an abelian group and F be a free abelian group generated by the symbols $u(m_1,m_2,\dots,m_n)$; $m_i\in M$, $i=1,2,\dots,n$. Let R be the subgroup of F generated by all elements of the type,

(a) $$\begin{aligned}&u(m_1,m_2,\dots,m_{i-1},m_im_{i+1},\dots,m_n)\\&-u(m_1,m_2,\dots,m_{i-1},m_i,m_{i+2},\dots,m_n)\\&-u(m_1,m_2,\dots,m_{i-1},m_{i+1},m_{1+2},\dots,m_n),\quad i=1,2,\dots,n\end{aligned}$$

and

(b) $$u(m_1,m_2,\dots,m_n)-u(m_{\pi(1)},m_{\pi(2)},\dots,m_{\pi(n)}).$$

where π is the permutation of the integers $1,2,\dots,n$. Then the nth symmetric product $\mathrm{Sp}^n(M)$ of M is defined to be the quotient group F/R. If we write $m_1\vee m_2\vee\cdots\vee m_n$ for the coset of $u(m_1,m_2,\dots,m_n)$, then a general element of $\mathrm{Sp}^n(M)$ is a finite sum of the form $\sum_{r\in Z} rm_1\vee m_2\vee\cdots\vee m_n$. Let G be a finite group

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such that $G = H \rtimes K$, $H \lhd G$. Let

$$\mathscr{H}: H = H_{(1)} \supseteq H_{(2)} = [H, G] \supseteq H_{(3)} = [H, G, G] \supseteq \cdots \supseteq H_{(m)} = [H, G, G, \ldots, G] \supseteq H_{(m+1)} = 1$$

be an N-series of H; a series of subgroups $H_{(i)}$ such that $[H_{(i)}, H_{(j)}] \subseteq H_{(i+j)}$ for all i, j. $\mathscr{H}$ induces a weight function w on H. For $x \in H$, $w(x) = k$ if $x \in H_{(k)} \setminus H_{(k+1)}$. $w(1) = \infty$. Define a family $\{\Lambda_m\}_{m=1}^{\infty}$ of Z-submodules of ZH as follows. Λ_k is spanned over Z by all products $(h_1 - 1)(h_2 - 1) \cdots (h_s - 1)$ with $\sum_{i=1}^{s} w(h_i) \geq k$. Then $\Lambda_0 = ZH$, $\Lambda_1 = A_H$ and $[\Lambda_{(i)}, \Lambda_{(j)}] \subseteq \Lambda_{(i+j)}$ for all $i, j \geq 0$. $\Lambda_i \supseteq A_H^i$ for all i. The filtration $\{\Lambda_k\}_{k=0}^{\infty}$ is called the canonical filtration of A_H with respect to $\mathscr{H}$. For $x \neq 1$, define $o^*(x)$ to be the order of the coset $xH_{(w(x)+1)}$. Since each of the quotient groups $H_{(m)}/H_{(m+1)}$ is finite abelian there exist elements $x_{i1}, x_{i2}, \ldots, x_{i\mu(i)}$ in $H_{(i)}/H_{(i+1)}$ such that any element $\bar{x} \in H_{(i)}/H_{(i+1)}$ can be written uniquely in the form

$$\bar{x} = b_1 \bar{x}_{i1} + b_2 \bar{x}_{i2} + \cdots + b_{\mu(i)} \bar{x}_{i\mu(i)},$$

where $0 \leq b_j \leq o^*(x_{ij})$ for all j, $1 \leq i \leq m$. Choose x_{ij} such that $o^*(x_{ij})$ divides $o^*(x_{ij+1})$. Set $S_0 = \{x_{ij} \mid i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, \mu(i)\}$. Order S_0 by putting $x_{ij} < x_{kl}$ if $i < k$ or $i = k$ and $j < l$. Enlarge S_0 to S by putting x_{ij}^{-1} immediately after x_{ij} if $o^*(x_{ij}) = \infty$. Let $|S| = n$. Re-index the set S by the integers $1, 2, \ldots, n$ so that $x_i < x_j$ if $i < j$. Then every element $h \in H$ can be written uniquely in the form

$$h = x_1^{e(1)} \cdot x_2^{e(2)} \cdots x_m^{e(m)} \tag{1}$$

where (1) $0 \leq e(i) \leq o^*(x_i)$ for all i, and

(2) if $x_{i+1} = x_i^{-1}$, then $e(i)e(i+1) = 0$.

The set S is then called the positive uniqueness basis of H.

With the above notations, an m-sequence $\alpha = (e(1), e(2), \ldots, e(m))$ is an ordered m-tuple of non-negative integers. The set S_m of all m-sequences is ordered lexicographically so that it is well-ordered. An m-sequence $\alpha = (e(1), e(2), \ldots, e(m))$ is basic if

(1) $0 \leq e(i) \leq d(i)$ for all i, $d(i) = o^*(x_i)$, and

(2) if $x_{i+1} = x_i^{-1}$, then $e(i)e(i+1) = 0$.

There is a one-to-one correspondence between the elements of H and the basic m-sequences. Define the weight $W(\alpha)$ of an m-sequence $\alpha = (e(1), e(2), \ldots, e(m))$ to be $W(\alpha) = \sum_{i=1}^{m} w(x_i)e(i)$. Define the proper product $P(\alpha) \in ZH$ to be $P(\alpha) = \prod_{i=1}^{m} (x_i - 1)^{e(i)}$ where the factors occur in order of increasing i from left to right. If α is basic, then $P(\alpha)$ is called a basic product.

3. Main results

Losey and Tahara found the Z-basis of $\Lambda_1, \Lambda_2, \Lambda_3$ and Λ_4 respectively. They are given by the following lemmas.

Lemma 1 ([2]). *The basic products form a free Z-basis of ZH. The basic products other than one form a free Z-basis of Δ_1.*

Lemma 2 ([2]). *Δ_2 has a free Z-basis consisting of*

(1) $(x_{1i}-1)^{d(i)}$, $d(i)=o^*(x_{1i})$,

(2) $P(\alpha)$, α *basic*, $W(\alpha)\geq 2$.

Lemma 3. *Δ_3 has a free Z-basis consisting of*

(1) $(x_{1i}-1)^{d(i)}$, $d(i)=o^*(x_{1i})\geq 3$,

(2) $(x_{2i}-1)^{d'(i)}$, $d'(i)=o^*(x_{2i})$,

(3) $d(i)(x_{1i}-1)(x_{1j}-1)$, $d(i)=o^*(x_{1i})$, $1\leq i\leq j\leq \mu(1)$,

(4) $P(\alpha)$, α *basic*, $W(\alpha)\geq 3$.

Lemma 4 ([4]). *Δ_4 has a system of Z-generators consisting of*

(1) $(x_{1i}-1)^{d(i)}$, $d(i)=o^*(x_{1i})\geq 4$,

(2) $(x_{2i}-1)^{d'(i)}$, $d'(i)=o^*(x_{2i})$,

(3) $(x_{3i}-1)^{d''(i)}$, $d''(i)=o^*(x_{3i})$,

(4) $(x_{1i}-1)^{d(i)}(x_{ij}-1)$, $d(i)=o^*(x_{1i})\geq 3$, $1\leq i\leq j\leq \mu(1)$,

(5) $(x_{1i}-1)(x_{1j}-1)^{d(j)}$, $d(j)=o^*(x_{1j})\geq 3$, $1\leq i\leq j\leq \mu(1)$,

(6) $(d(i),d'(j))(x_{1i}-1)(x_{2j}-1)$, $d(i)=o^*(x_{1i})$, $d'(j)=o^*(x_{2j})$,

(7) $d(i)(x_{1i}-1)(x_{1j}-1)(x_{1k}-1)$, $d(i)=o^*(x_{1i})$, $1\leq i\leq j\leq k\leq \mu(1)$,

(8) $P(\alpha)$, α *basic*, $W(\alpha)\geq 4$.

Denote by $W_2(K)=(K_2/K_3)\oplus \mathrm{Sp}^2(K_1/K_2)$ and $\mathscr{W}_2(H)=(H_{(2)}/H_{(3)})\oplus \mathrm{Sp}^2(H_{(1)}/H_{(2)})$. We now prove the following lemma.

Lemma 5. *Let G be a finite group and H a normal subgroup of G such that $G=H\,]\,K$. Then*

$$\begin{aligned}\Delta_G^2\Delta_H/\Delta_G^3\Delta_H &= (\Delta_H^3+\Delta_{[H,K]}\Delta_H)/(\Delta_H^4+\Delta_H\Delta_{[H,K]}\Delta_H+\Delta_{[H,K,H]}\Delta_H+\Delta_{[H,K,K]}\Delta_H)\\ &\oplus(\Delta_K\Delta_H^2+\Delta_K^2\Delta_H)/(\Delta_K\Delta_H^3+\Delta_K^2\Delta_H^2+\Delta_K^3\Delta_H+\Delta_K\Delta_{[H,K]}\Delta_H).\end{aligned}$$

Proof. Δ_G is freely generated as an abelian group by the set $\{g-1\mid g\in G\}$.

$$A_G = \langle g-1 \mid g \in G\rangle,$$

$$g-1 = hk-1; \qquad h\in H, \quad k \in K,$$

$$= (h-1)(k-1) + (h-1) + (k-1).$$

So

$$A_G = A_H + A_K + A_H A_K.$$

By (1) this sum is a direct sum.

$$A_G A_H = A_H^2 + A_K A_H + A_H A_K A_H,$$

$$A_G^2 A_H = (A_H + A_K + A_H A_K)(A_H^2 + A_K A_H + A_H A_K A_H)$$

$$= A_H^3 + A_K A_H^2 + A_H A_K A_H^2 + A_H A_K A_H + A_K^2 A_H$$

$$+ A_H A_K^2 A_H + A_H^2 A_K A_H + A_K A_H A_K A_H + A_H A_K A_H A_K A_H$$

$$= A_H^3 + A_K A_H^2 + A_H A_K A_H + A_K^2 A_H + A_K A_H A_K A_H$$

$$+ A_H A_K A_H A_K A_H \tag{1}$$

Since

$$A_H A_K \subseteq A_K A_H + A_H, \qquad A_H A_K A_H \subseteq A_K A_H^2 + A_H^2$$

and

$$A_K A_H A_K A_H \subseteq A_K(A_K A_H^2 + A_H^2) \subseteq A_K A_H^2,$$

we have

$$A_H A_K A_H A_K A_H \subseteq A_H(A_K A_H^2) \subseteq A_H A_K A_H.$$

From (1)

$$A_G^2 A_H = A_H^3 + A_K A_H^2 + A_K^2 A_H + A_H A_K A_H, \tag{2}$$

$$A_G^3 A_H = (A_H + A_K + A_H A_K)(A_H^3 + A_K A_H^2 + A_K^2 A_H + A_H A_K A_H)$$

$$= A_H^4 + A_K A_H^3 + A_K^2 A_H^2 + A_K^3 A_H + A_H A_K A_H^2 + A_H A_K^2 A_H$$

$$+ A_H^2 A_K A_H + A_K A_H A_K A_H + A_H A_K A_H A_K A_H, \tag{3}$$

$$A_H A_K A_H^3 \subseteq A_H A_K A_H^2, \qquad A_H A_K^2 A_H^2 \subseteq A_H A_K^2 A_H$$

and

$$A_H A_K^3 A_H \subseteq A_H A_K^2 A_H.$$

Consider the identity

$$(y-1)(x-1) = (x-1)(y-1) + ([y,x]-1) + (x-1)([y,x]-1)$$

$$+ (y-1)([y,x]-1) + (x-1)(y-1)([y,x]-1).$$

So for $h \in H$, $k \in K$,

$$(h-1)(k-1) = (k-1)(h-1) + ([h,k]-1) + (k-1)([h,k]-1)$$

$$+ (h-1)([h,k]-1) + (k-1)(h-1)([h,k]-1)$$

$$(k-1)([h,k]-1)=(k-1)(h^{-1}h^k-1)$$
$$=(k-1)[(h^{-1}-1)(h^k-1)+(h^{-1}-1)+(h^k-1)]$$
$$\subseteq A_K A_H^2 + A_K A_H \subseteq A_K A_H.$$

Also,

$$(k-1)(h-1)([h,k]-1)$$
$$=(k-1)(h-1)(h^{-1}h^k-1)$$
$$=(k-1)(h-1)[(h^{-1}-1)(h^k-1)+(h^{-1}-1)+(h^k-1)]$$
$$\subseteq A_K A_H^3 + A_K A_H^2 \subseteq A_K A_H.$$

Hence

$$A_H A_K \subseteq A_K A_H + A_{[H,K]} + A_H A_{[H,K]}.$$

So

$$A_H A_K A_H \subseteq (A_K A_H + A_{[H,K]} + A_H A_{[H,K]}) A_H$$
$$\subseteq A_K A_H^3 + A_{[H,K]} A_H + A_H^3. \tag{5}$$

Thus

$$A_H A_K A_H^2 \subseteq (A_K A_H^2 + A_{[H,K]} A_H + A_H^3) A_H$$
$$\subseteq A_K A_H^2 + A_{[H,K]} A_H^2 + A_H^4.$$

Let $x \in A_{[H,K]} A_H^2$ be such that $x=(a-1)(h_1-1)(h_2-1)$, $a \in [H,K]$; $h_1, h_2 \in H$. Then

$$x \in (A_H A_{[H,K]} + A_{[H,K,H]} + A_{[H,K]} A_{[H,K,H]}) A_H.$$
$$\in A_H A_{[H,K]} A_H + A_{[H,K,H]} A_H.$$

Hence

$$A_H A_K A_H^2 \subseteq A_K A_H^3 + A_H A_{[H,K]} A_H + A_{[H,K,H]} A_H + A_H^4. \tag{7}$$

$$A_H^2 A_K A_H \subseteq A_H (A_K A_H^2 + A_{[H,K]} A_H + A_H^3)$$
$$\subseteq A_H A_K A_H^2 + A_H A_{[H,K]} A_H + A_H^4$$
$$\subseteq A_K A_H^3 + A_H A_{[H,K]} A_H + A_{[H,K,H]} A_H + A_H^4. \tag{8}$$

$$A_K A_H A_K A_H \subseteq A_K (A_K A_H^2 + A_{[H,K]} A_H + A_H^3)$$
$$\subseteq A_K^2 A_H^2 + A_K A_{[H,K]} A_H + A_K A_H^3. \tag{9}$$

$$A_H A_K A_H A_K A_H \subseteq A_H (A_K^2 A_H^2 + A_K A_{[H,K]} A_H + A_K A_H^3)$$
$$\subseteq A_H A_K^2 A_H + A_H A_K A_H^2. \tag{10}$$

$$A_H A_K^2 A_H = A_H A_K A_K A_H$$
$$\subseteq (A_K A_H + A_{[H,K]} + A_H A_{[H,K]})) A_K A_H$$
$$\subseteq A_K A_H A_K A_H + A_{[H,K]} A_K A_H + A_H A_{[H,K]} A_K A_H.$$

Hence

$$A_HA_KA_HA_KA_H \subseteq A_K^2A_H^2 + A_KA_H^3 + A_{[H,K,K]}A_H + A_HA_KA_H^2 + A_H^2A_KA_H$$
$$\subseteq A_K^2A_H^2 + A_KA_H^3 + A_{[H,K,K]}A_H$$
$$+ A_{[H,K,H]}A_H + A_HA_{[H,K]}A_H + A_H^4. \tag{11}$$

From (3) using (7), (8), (9)

$$A_G^3A_H = A_H^4 + A_KA_H^3 + A_K^2A_H^2 + A_K^3A_H + A_HA_{[H,K]}A_H$$
$$+ A_{[H,K,K]}A_H + A_{[H,K,H]}A_H + A_KA_{[H,K]}A_H.$$

From (2) using (5),

$$A_G^2A_H = A_H^3 + A_KA_H^2 + A_K^2A_H + A_HA_KA_H.$$

But $A_HA_KA_H \subseteq A_G^2A_H$, therefore we have equality in the last line. Thus

$$A_G^2A_H = (A_H^3 + A_{[H,K]}A_H) \oplus (A_KA_H^2 + A_K^2A_H)$$

and

$$A_G^3A_H = (A_H^4 + A_HA_{[H,K]}A_H + A_{[H,K,H]}A_H + A_{[H,K,K]}A_H)$$
$$\oplus (A_KA_H^3 + A_K^2A_H^2 + A_K^3A_H + A_KA_{[H,K]}A_H).$$

Therefore

$$A_G^2A_H/A_G^3A_H$$
$$\simeq (A_H^3 + A_{[H,K]}A_H)/(A_H^4 + A_HA_{[H,K]}A_H + A_{[H,K,H]}A_H + A_{[H,K,K]}A_H)$$
$$\oplus (A_KA_H^2 + A_K^2A_H)/(A_KA_H^3 + A_K^2A_H^2 + A_K^3A_H + A_KA_{[H,K]}A_H).$$

This completes the proof of the lemma.

We determine the structure of the two direct summands on the right hand side separately. The following lemma gives the structure of the first term completely.

Lemma 6. *There exists a homomorphism*

$$\psi^*: A_3^*/A_4^* \twoheadrightarrow W_3^*/R_3^*$$

whose kernel is

$$(A_{[H,K,K]} + A_{[H,K,H]} + A_{H_3} + A_{[H,K]}^2) \cap A_3^* + A_4^*$$

where $A_3^* = A_H^3 + A_{[H,K]}A_H$ *and*

$$A_4^* = A_H^4 + A_HA_{[H,K]}A_H + A_{[H,K,H]}A_H + A_{[H,K,K]}A_H.$$
$$W_3^* = \mathrm{Sp}^3(H_{(1)}/H_{(2)}) \oplus (H_{(1)}/H_{(2)} \otimes H_{(2)}/H_{(3)})$$

and R_3^* is the subgroup of W_3^* generated by elements

$$\frac{d(j)}{d(i)}(\bar{x}_{1j}\otimes\overline{x_{1i}^{d(i)}})-(\bar{x}_{1i}\otimes\overline{x_{1j}^{d(j)}})+\binom{d(j)}{2}(\bar{x}_{1i}\vee\bar{x}_{1j}\vee\bar{x}_{1k})$$
$$-\frac{d(j)}{d(i)}\binom{d(i)}{2}(\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1j}),\qquad 1\le i\le j\le\lambda(1).$$

Proof. Define ψ on the Z-free generators of Λ_3 as follows:

$$(x_{1i}-1)^{d(i)}\psi=\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1i}\quad\text{if } d(i)=3,$$
$$=R_3^*\qquad\text{if } d(i)>3.$$
$$(x_{2i}-1)^{d'(i)}\psi=R_3^*.$$
$$d(i)(x_{1i}-1)(x_{1j}-1)\psi=-\binom{d(i)}{2}(\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1j})$$
$$+\bar{x}_{1j}\otimes\overline{x_{1i}^{d(i)}}+R_3^*\quad\text{where } d(i)=o^*(x_{1i}).$$
$$(x_{1i}-1)(x_{1j}-1)(x_{1k}-1)\psi=\bar{x}_{1i}\vee\bar{x}_{1j}\vee\bar{x}_{1k}+R_3^*,\quad 1\le i\le j\le k\le\mu(1).$$
$$(x_{1i}-1)(x_{2j}-1)\psi=\bar{x}_{1i}\otimes\bar{x}_{2j}+R_3^*.$$
$$(x_{3i}-1)\psi=R_3^*.$$

$(P(\alpha))\psi=R_3^*$ where α is basic and $W(\alpha)\ge 4$. $\Lambda_3^*\subset\Lambda_3$ since it is spanned by elements $(h_1-1)(h_2-1)(h_3-1),(x-1)(y-1)$; $w(h_i)=1$, $x\in[H,K]$ and of weight 2, and $y\in H$. So $\sum_{i=1}^3 w(h_i)=3$ and $wt\cdot x+wty=3$. Similarly $\Lambda_4^*\subset\Lambda_4$. Therefore ψ induces a homomorphism $\psi^*:\Lambda_3^*\to W_3^*/R_3^*$. We now show that $(\Lambda_4)\psi^*=R_3^*$ so that ψ^* actually induces a homomorphism $\psi^*:\Lambda_3^*/\Lambda_4^*\to W_3^*/R_3^*$.

Consider the image of ψ^* on each of the basis elements of Λ_4.

$$(x_{1i}-1)^{d(i)}\psi^*=R_3^*\qquad\text{since } d(i)\ge 4.$$
$$(x_{2i}-1)^{d'(i)}\psi^*=R_3^*,\quad d'(i)=o^*(x_{2i}).$$
$$(x_{3i}-1)^{d''(i)}\psi^*=R_3^*,\quad d''(i)=o^*(x_{3i}).$$
$$(x_{1i}-1)^{d(i)}(x_{1j}-1)\psi^*=\Bigg[-d(i)(x_{1i}-1)(x_{1j}-1)-\binom{d(i)}{2}(x_{1i}-1)^2(x_{1j}-1)$$
$$-\sum_{k=3}^{d(i)-1}\binom{d(i)}{2}(x_{1i}-1)^k(x_{1j}-1)+(x_{1i}^{d(i)}-1)(x_{1j}-1)\Bigg]\psi^*$$
$$=\Bigg[-d(i)(x_{1i}-1)(x_{1j}-1)-\binom{d(i)}{2}(x_{1i}-1)^2(x_{1j}-1)$$
$$-\sum_{k=3}^{d(i)-1}\binom{d(i)}{k}(x_{1i}-1)^k(x_{1j}-1)+(x_{1j}-1)(x_{1i}^{d(i)}-1)$$

$$+([x_{1i}^{d(i)}, x_{1j}] - 1) + \sum g(\alpha)P(\alpha)\Big]\psi^*$$

$$= \binom{d(i)}{2}(\bar{x}_{1i} \vee \bar{x}_{1i} \vee \bar{x}_{1i}) - (\bar{x}_{1j} \otimes \overline{x_{1i}^{d(i)}})$$

$$- \binom{d(i)}{2}(\bar{x}_{1i} \vee \bar{x}_{1i} \vee \bar{x}_{1i}) + (\bar{x}_{1j} \otimes \overline{x_{1i}^{d(i)}}) + R_3^* = R_3^*.$$

$$[(x_{1i}-1)(x_{1j}-1)^{d(j)}]\psi^* = \Big[-d(j)(x_{1i}-1)(x_{1j}-1) - \binom{d(j)}{2}(x_{1i}-1)(x_{1j}-1)^2$$

$$- \sum_{k=3}^{d(j)-1} \binom{d(j)}{k}(x_{1i}-1)(x_{1j}-1)^k + (x_{1i}-1)(x_{1j}^{d(j)}-1)\Big]\psi^*$$

$$= \binom{d(i)}{2}\frac{d(j)}{d(i)}(\bar{x}_{1i} \vee \bar{x}_{1i} \vee \bar{x}_{1i}) - \frac{d(j)}{d(i)}(\bar{x}_{1j} \otimes \overline{x_{1i}^{d(i)}})$$

$$- \binom{d(j)}{2}(\bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1j}) + (\bar{x}_{1i} \otimes \overline{x_{1j}^{d(j)}}) + R_3^* = R_3^*.$$

$$(d(i), d'(j))(x_{1i}-1)(x_{2j}-1)\psi^* = (d(i), d'(j))(\bar{x}_{1i} \otimes \bar{x}_{2j})$$

$$= \bar{x}_{1i} \otimes \overline{x_{2j}^{d(j)}} = R_3^*.$$

$$[d(i)(x_{1i}-1)(x_{1j}-1)(x_{1k}-1)]\psi^* = d(i)(\bar{x}_{1i} \vee \bar{x}_{1j} \vee \bar{x}_{1k})$$

$$= \text{identity in } \mathrm{Sp}^3(H_{(1)}/H_{(2)}).$$

$$[P(\alpha)]\psi^* = R_3^*$$

by definition where α is basic and $W(\alpha) \geq 4$. Hence

$$(A_4)\psi^* = R_3^*.$$

ψ^* is clearly onto by definition. To determine Ker ψ^*, if $x \in A_3$ is expressed as a linear combination of its Z-free generators, we can observe that $x\psi^* = 0$ implies that $x = 0$. But by definition of ψ, the elements of the type, $x_{3i} - 1$; $(x_{2i}-1)^{d'(i)}$; $P(\alpha)$, α basic $W(\alpha) \geq 4$ are mapped into R_3^*. Therefore these elements lie in Ker ψ^*. These elements are precisely

$$(A_{[H,K,K]} + A_{[H,K,H]} + A_{H_3} + A^2_{[H,K]}) \cap A_3^*.$$

Hence

$$\text{Ker } \psi^* = (A_{[H,K,K]} + A_{[H,K,H]} + A_{H_3} + A^2_{[H,K]}) \cap A_3^* + A_4^*.$$

This completes the proof of the lemma.

Lemma 7

$$\frac{A_K^2 A_H + A_K A_H^2}{A_K^3 A_H + A_K^2 A_H^2 + A_K A_H^3} \simeq \frac{[W_2(K) \otimes (H_{(1)}/H_{(2)})] \oplus [(K_1/K_2) \otimes \mathscr{W}_2(H)]}{R_3(H,K)}$$

where $R_3(H,K)$ is the subgroup of $[W_2(K) \otimes (H_{(1)}/H_{(2)})] \oplus [(K_1/K_2) \otimes \mathscr{W}_2(H)]$ generated by elements m_{ij} and n_{ij} where $m_{ij} = \bar{y}_i^m \otimes \bar{x}_j - \bar{y}_i \otimes \bar{x}_j^m$, $m = [d(i), d'(i)]$ is the least common multiple of $d(i) = o^*(x_i)$ and $d'(i) = o^*(y_i)$;

$$n_{ij}=\frac{[d(j),d'(i)]}{d'(i)}\left\{\left[\overline{y_{1i}^{d'(i)}}-\binom{d'(i)}{2}(\bar{y}_i\vee\bar{y}_i)\right]\otimes\bar{x}_j\right\}$$

$$-\frac{[(d(j),d'(i)]}{d(j)}\left\{\bar{y}_i\otimes\left[\overline{x_j^{d(j)}}-\binom{d(j)}{2}(\bar{x}_j\vee\bar{x}_j)\right]\right\}$$

for $1\leq i\leq\lambda;\quad 1\leq j\leq\mu.$

Proof. Denote $M_1=A_K^2A_H+A_KA_H^2$ and $M_2=A_K^3A_H+A_K^2A_H^2+A_KA_H^3$. Define a mapping $\theta_1:A_K^2\times A_H\to M_1/M_2$ by $(u_2,v)\theta_1=u_2v+M_2$ where $u_2\in A_K^2$ and $v\in A_H$. We prove that (1) $(A_K^3\times A_H)\theta_1=M_2$ and (2) $(A_K^2\times A_H^2)\theta_1=M_2$. A_K^3 is generated additively by $(k_1-1)(k_2-1)(k_3-1)$; $k_1,k_2,k_3\in K$.

For $x=(k_1-1)(k_2-1)(k_3-1)\in A_K^3$, if $h\in H$,

$$\begin{aligned}(x,h-1)\theta_1&=[(k_1k_2-1)(k_3-1)-(k_1-1)(k_3-1)-(k_2-1)(k_3-1)](h-1)\theta_1\\&=(k_1-1)(k_2-1)(k_3-1)(h-1)\in A_K^3A_H\subset M_2.\end{aligned}$$

A_K^2 is generated additively by $(k_1-1)(k_2-1)$; $k_1,k_2\in K$. Similarly A_H^2 is generated additively by $(h_1-1)(h_2-1)$; $h_1,h_2\in H$.

Let $x=(k_1-1)(k_2-1)\in A_K^2$ and $y=(h_1-1)(h_2-1)$ be an element of A_H^2.

$$\begin{aligned}(x,y)\theta_1&=x\cdot y+M_2\\&=(k_1-1)(k_2-1)[(h_1h_2-1)+(h_1-1)+(h_2-1)]+M_2\\&=(k_1-1)(k_2-1)(h_1-1)(h_2-1)\in A_K^2A_H^2\subset M_2.\end{aligned}$$

Hence θ_1 induces a mapping $\hat{\theta}_1:(A_K^2/A_K^3)\times(A_H/A_H^2)\to M_1/M_2$ defined by $(\bar{u}_2,\bar{v})\hat{\theta}_1=u_2v+M_2$ where $\bar{u}_2=u_2+A_K^3$, $u_2\in A_K^2$, $\bar{v}=v+A_H^2$, $v\in A_H$.

It is easy to prove that $\hat{\theta}_1$ is bilinear.

$$\begin{aligned}(\bar{u}_2+\bar{u}_2',\bar{v})\hat{\theta}_1&=(\overline{u_2+u_2'},\bar{v})\hat{\theta}_1=(u_2+u_2')\cdot v+M_2=u_2v+u_2'v+M_2\\&=(\bar{u}_2,\bar{v})\hat{\theta}_1+(\bar{u}_2',\bar{v})\hat{\theta}_1\end{aligned}$$

Similarly

$$(\bar{u}_2,\overline{v_1+v_2})\hat{\theta}_1=u_2\cdot(v_1+v_2)+M_2=u_2v_1+u_2v_2+M_2.$$

Therefore $\hat{\theta}_1$ induces a homomorphism

$$\hat{\hat{\theta}}_1:(A_K^2/A_K^3)\otimes(A_H/A_H^2)\to M_1/M_2$$

given by $(\bar{u}_2\otimes\bar{v})\hat{\hat{\theta}}_1=u_2v+M_2$.

Define $\theta_2:A_K\times A_H^2\to M_1/M_2$ by $(u,v)\theta_2=uv+M_2$; $u\in A_K$, $v\in A_H^2$. We can easily prove that (1) $(A_K^2\times A_H^2)\theta_2=M_2$ and (2) $(A_K\times A_H^3)\theta_2=M_2$. Therefore θ_2 induces a bilinear mapping $\hat{\theta}_2:(A_K/A_K^2)\times(A_H^2/A_H^3)\to M_1/M_2$ inducing a homomorphism $\hat{\hat{\theta}}_2:(A_K/A_K^2)\otimes(A_H^2/A_H^3)\to M_1/M_2$ given by $(\bar{u}\otimes\bar{v}_2)\hat{\hat{\theta}}_2=uv_2+M_2$. Thus there exists a homomorphism $\hat{\hat{\theta}}=\hat{\hat{\theta}}_1+\hat{\hat{\theta}}_2$ from $[(A_K^2/A_K^3)\otimes(A_H/A_H^2)]\oplus[(A_K/A_K^2)\otimes(A_H^2/A_H^3)]$ to M_1/M_2 defined by

$$(u_2\otimes\bar{v}+\bar{u}\otimes\bar{v}_2)\hat{\hat{\theta}}=u_2v+uv_2+M_2.$$

Since [2], $A_K^2/A_K^3 \simeq W_2(K)$, $A_H/A_H^2 \simeq H_1/H_2$, $A_K/A_K^2 \simeq K_1/K_2$ and $A_H^2/A_H^3 \simeq \mathscr{W}_2(H)$, we have a homomorphism

$$\tilde{\theta}: [W_2(K)\otimes(H_{(1)}/H_{(2)})]\oplus[(K_1/K_2)\otimes \mathscr{W}_2(H)]\to M_1/M_2$$

defined by

$$(\bar{y}_2\otimes\bar{x}_1+(\bar{y}_1\vee\bar{y}_1')\otimes\bar{x}_1'+\bar{y}_1''\otimes\bar{x}_2+\overline{y_1'''}\otimes(\bar{x}_1''\vee\bar{x}_1'''))\tilde{\theta}$$
$$=(y_2-1)(x_1-1)+(y_1-1)(y_1'-1)(x_1'-1)+(y_1''-1)(x_2-1)$$
$$+(y_1'''-1)(x_1''-1)(x_1'''-1)+M_2,$$

where $y_1, y_1', y_1'', y_1'''\in K_1$, $y_2\in K_2$; $x_1, x_1', x_1'', x_1'''\in H_{(1)}$; $x_2\in H_{(2)}$.

We prove that $R_3(H,K)\tilde{\theta}=M_2$. For m_{ij}, $n_{ij}\in R_3(H,K)$.

$$m_{ij}\tilde{\theta}=(\bar{y}_i'''\otimes\bar{x}_j-\bar{y}_i\otimes\bar{x}_j''')\tilde{\theta}$$
$$=(y_i'''-1)(x_j-1)-(y_i-1)(x_j'''-1)+M_2$$
$$=m[(y_i-1)(x_j-1)-(y_i-1)(x_j-1)]\quad \text{modulo } M_2$$
$$=M_2.$$

$$(n_{ij})\tilde{\theta}=\frac{[d(j),d'(i)]}{d'(i)}\left[(y_i^{d'(i)}-1)(x_j-1)-\binom{d'(i)}{2}(y_i-1)(y_i-1)(x_j-1)\right]$$
$$-\frac{[(d(j),d'(i)]}{d(j)}\left[(y_i-1)(x_j^{d(j)}-1)-\binom{d(j)}{2}(y_i-1)(x_j-1)(x_j-1)\right]$$
$$=\frac{[d(j),d'(i)]}{d'(i)}\left[d'(i)(y_i-1)+\sum_{k=3}^{d'(i)-1}\binom{d'(i)}{k}(y_i-1)^k\right](x_j-1)$$
$$-\frac{[d(j),d'(i)]}{d(j)}(y_i-1)\left[d(j)(x_j-1)+\sum_{l=3}^{d(j)-1}\binom{d(j)}{l}(x_j-1)^l\right]+M_2$$
$$=M_2.$$

Let

$$W=[W_2(K)\otimes(H_{(1)}/H_{(2)})]\oplus[(K_1/K_2)\otimes \mathscr{W}_2(H)].$$

Then $\tilde{\theta}$ induces a homomorphism $\theta: W/R_3(H,K)\to M_1/M_2$. We construct a homomorphism $\sigma: M_1/M_2\to W/R_3(H,K)$ such that $\theta\sigma=$ identity on $W/R_3(H,K)$ and $\sigma\theta=$ identity on M_1/M_2.

Let $T=\{y_{ij};\ i=1,2,\ldots,m;\ j=1,2,\ldots,\lambda\}$ be the positive uniqueness basis of K. For $x\in H$, define $\sigma_1: A_K^2 A_H\to W/R_3(H,K)$ using the Z-free generators of A_K^2 as follows:

$$d'(i)(y_{1i}-1)(x-1)\sigma_1=\left(\bar{y}_{1i}^{d'(i)}-\binom{d'(i)}{2}(\bar{y}_{1i}\vee\bar{y}_{1i})\right)\otimes\bar{x}+R_3(H,K),$$
$$1\le i\le\lambda,\qquad d'(i)=o^*(y_{1i}).$$
$$(y_{1i}-1)(y_{1j}-1)(x-1)\sigma_1=(\bar{y}_{1i}\vee\bar{y}_{1j})\otimes\bar{x}+R_3(H,K),\quad 1\le i\le j\le\lambda.$$
$$(y_{2i}-1)(x-1)\sigma_1=\bar{y}_{2i}\otimes\bar{x}+R_3(H,K).$$
$$P(\alpha)(x-1)\sigma_1=R_3(H,K),\quad W(\alpha)\ge 3.$$

Then $(A_K^2 A_H^2)\sigma_1 = R_3(H,K)$ using the Z-free generators $\{(h_1-1)(h_2-1)\,|\,h_1,h_2\in H\}$ of A_H^2. To prove that $(A_K^3 A_H)\sigma_1 = R_3(H,K)$ consider the free Z-generators of A_K^3 consisting of

$$(y_{1i}-1)^{d'(i)},\quad d'(i)=o^*(y_{1i})\geq 3;$$

$$(y_{2i}-1)^{d''(i)},\quad d''(i)=o^*(y_{2i});$$

$$d'(i)(y_{1i}-1)(y_{1j}-1),\quad 1\leq i\leq j\leq \lambda;$$

$$(y_{1i}-1)(y_{1j}-1)(y_{1k}-1),\quad 1\leq i\leq j\leq k\leq \lambda;$$

and $P(\alpha)$ with α basic and $W(\alpha)\geq 3$. We consider the image of σ_1 on each of the basis elements.

$$\begin{aligned}[(y_{1i}-1)^{d'(i)}(h-1)]\sigma_1 &= \left[(y_{1i}^{d'(i)}-1)-d'(i)(y_{1i}-1)-\binom{d'(i)}{2}(y_{1i}-1)^2\right.\\ &\qquad\left.-\sum_{k=3}^{d'(i)-1}\binom{d'(i)}{k}(y_{1i}-1)^k\right](h-1)\sigma_1\\ &= y_{1i}^{d'(i)}\otimes\bar h - y_{1i}^{d'(i)}\otimes\bar h+\binom{d'(i)}{2}(\bar y_{1i}\vee\bar y_{1i})\otimes\bar h\\ &\qquad-\binom{d'(i)}{2}(\bar y_{1i}\vee\bar y_{1i})\otimes\bar h = R_3(H,K).\end{aligned}$$

$$\begin{aligned}[(y_{2i}-1)^{d''(i)}(h-1)]\sigma_1 &= \left[-d''(i)(y_{2i}-1)-\binom{d''(i)}{2}(y_{2i}-1)^2\right.\\ &\qquad\left.-\sum_{k=3}^{d''(i)-1}\binom{d''(i)}{k}(y_{2i}-1)^k+(y_{2i}^{d''(i)}-1)\right](h-1)\sigma_1\end{aligned}$$

The last two terms

$$\sum_{k=3}^{d''(i)-1}\binom{d''(i)}{2}(y_{2i}-1)^k(h-1)\quad\text{and}\quad(y_{2i}^{d''(i)}-1)(h-1)$$

on the right hand side lie in $P(\alpha)$, α basic with $W(\alpha)\geq 3$, so their image lies in $R_3(H,K)$. Also

$$-\binom{d''(i)}{2}(y_{2i}-1)^2(h-1)\sigma_1\in R_3(H,K)$$

since the weight of $(y_{2i}-1)^2=4$. So

$$\begin{aligned}[(y_{2i}-1)^{d''(i)}(h-1)]\sigma_1 &= [-d''(i)(y_{2i}-1)(h-1)]\sigma_1\\ &= -d''(i)(\bar y_{2i}\otimes\bar h)\in R_3(H,K).\end{aligned}$$

$$\begin{aligned}[d'(i)(y_{1i}-1)(y_{1i}-1)(h-1)]\sigma_1 &= d'(i)(\bar y_{1i}\vee\bar y_{1j})\otimes\bar h+R_3(H,K)\\ &= R_3(H,K),\end{aligned}$$

$(y_{1i}-1)(y_{1j}-1)(y_{1k}-1)(h-1)\in P(\alpha)$ with α basic and $W(\alpha)\geq 3$ so its image under σ_1 lies in $R_3(H,K)$.

Similarly we define a homomorphism $\sigma_2: A_K A_H^2 \to W/R_3(H,K)$ as follows. For any $k\in K$,

$$d(i)(k-1)(x_{1i}-1)\sigma_2=\bar{k}\otimes\left(\overline{x_{1i}^{d(i)}}-\binom{d(i)}{2}(\bar{x}_{1i}\vee\bar{x}_{1i})\right)+R_3(H,K),$$

$$(k-1)(x_{1i}-1)(x_{1j}-1)\sigma_2=\bar{k}\otimes(\bar{x}_{1i}\vee\bar{x}_{1j})+R_3(H,K),\quad 1\leq i\leq j\leq\mu,$$

$$(k-1)P(\beta)\sigma_2=R_3(H,K),\quad W(\beta)\geq 3.$$

It is easy to prove that $(A_K A_H^3)\sigma_2=R_3(H,K)$. By [5],

$$A_K^2 A_H\cap A_K A_H^2=\langle[d'(i),d(j)](y_i-1)(x_j-1)\mid 1\leq i\leq\lambda,\ 1\leq j\leq\mu\rangle$$

modulo M_2. We prove that σ_1 and σ_2 map any element of $A_K^2 A_H\cap A_K A_H^2$ to the same image. For,

$$\begin{aligned}&[d'(i),d(j)](y_i-1)(x_j-1)\sigma_1\\&=\frac{[d'(i),d(j)]}{d'(i)}\left[\left(\bar{y}_i^{d'(i)}-\binom{d'(i)}{2}(\bar{y}_i\vee\bar{y}_i)\right)\otimes\bar{x}_j\right]+R_3(H,K)\\&=\frac{[d'(i),d(j)]}{d(j)}\left\{\bar{y}_i\otimes\left[\bar{x}_j^{d(j)}-\binom{d(j)}{2}(\bar{x}_j\vee\bar{x}_j)\right]\right\}\\&=[d'(i),d(j)](y_i-1)(x_j-1)\sigma_2.\end{aligned}$$

Therefore we have a homomorphism

$$\sigma=\sigma_1+\sigma_2:\frac{(A_K^2 A_H+A_K A_H^2)}{(A_K^3 A_H+A_K^2 A_H^2+A_K A_H^3)}\to W/R_3(H,K)$$

induced by σ_1 and σ_2. It is easy to verify that $\bar{\theta}\sigma$ is the identity map on $W/R_3(H,K)$ and $\sigma\bar{\theta}$ is the identity map on M_1/M_2.

Therefore $M_1/M_2\simeq W/R_3(H,K)$.

Note ([1]). Lemma 3.6 immediately gives us

$$\mathrm{Sp}^2(H/[H,K])\simeq\mathrm{Sp}^2(H/[H,G])/\mathrm{Im}([H,K]\vee H).$$

Theorem 8. *Let G be a finite group such that $G=H\rfloor K$, a split extension of a normal subgroup H by a subgroup K, then*

$$\begin{aligned}\frac{A_K^2 A_H+A_K A_H^2}{A_K^3 A_H+A_K^2 A_H^2+A_K A_H^3+A_K A_{[H,K]}A_H}&\simeq[K_2/K_3\otimes H_1/H_2]\\&\oplus[\mathrm{Sp}^2(K_1/K_2)\otimes(H_1/H_2)]\oplus[(K_1/K_2)\otimes([H,K]/H_{(3)})]\\&\oplus[(K_1/K_2)\otimes(\mathrm{Sp}^2[H,K]/[H,G])].\end{aligned}$$

Proof. Consider the following diagram.

$$
\begin{array}{ccc}
0 & & 0 \\
\uparrow & & \uparrow \\
\dfrac{A_K^2A_H + A_KA_H^2}{A_K^3A_H + A_K^2A_H^2 + A_KA_H^3 + A_KA_{[H,K]}A_H} & \xrightarrow{\bar{\sigma}} & \begin{array}{l}[W_2(K)\otimes(H_1/H_2)] \\ \oplus[(K_1/K_2)\otimes([H,K]/H_{(3)})] \\ \oplus(K_1/K_2)\otimes \mathrm{Sp}^2(H/H_{(2)})]\end{array} \\
\uparrow & & \uparrow \\
\dfrac{A_K^2A_H + A_KA_H^2}{A_K^3A_H + A_K^2A_H^2 + A_KA_H^3} & \underset{\bar{\theta}}{\overset{\sigma}{\rightleftarrows}} & W/R_3(H,K) \\
\uparrow & & \uparrow \\
\dfrac{A_K^3A_H + A_K^2A_H^2 + A_KA_H^3 + A_KA_{[H,K]}A_H}{A_K^3A_H + A_K^2A_H^2 + A_KA_H^3} & \underset{\psi}{\overset{\sigma'}{\rightleftarrows}} & \begin{array}{l}\{[(K_1/K_2)\otimes(H_2/H_{(3)})] \\ \oplus[(K_1/K_2)\otimes([H,K]\vee H)] \\ /R_3(H,K)\}\end{array} \\
\uparrow & & \\
0 & &
\end{array}
$$

The columns are exact.

$\bar{\sigma}$ is induced by σ if $(A_KA_{[H,K]}A_H)\sigma$ lies in the image of

$$[(K_1/K_2)\otimes(H_2/H_{(3)})]\oplus((K_1/K_2)\otimes([H,K]\vee H))$$

Let $a\in[H,K]$. Let $x_1,x_2,\dots,x_r,x_1',x_2',\dots,x_s'$ be a positive uniqueness basis of H where $x_i's$ are of weight one and $x_j's$ are of weight 2.

Let $a=a_0a_1$ where a_0 is a product of x_is and a_1 is a product of x_js. Then $y=(k-1)(a_0a_1-1)(h-1)$ is an element of $A_KA_{[H,K]}A_H$ with $h\in H$, $k\in K$.

$$
\begin{aligned}
y&=(k-1)[(a_0-1)(a-1)+(a_0-1)+(a-1)](h-1)\\
&\equiv(k-1)(a_0-1)(h-1) \quad \text{modulo } A_KA_H^3.
\end{aligned}
$$

Let $a_0=\prod_{i=1}^r x_i^{p_i}$; p_is are integers. W.l.o.g. we can assume that $h=x_t$ for some t, $1\le t\le r$.

$$
\begin{aligned}
y&=\sum_{i=1}^{r} p_i(k-1)(x_i-1)(x_t-1)\\
&=\sum_{i=1}^{t} p_i(k-1)(x_i-1)(x_t-1)+\sum_{i=t+1}^{r} p_i(k-1)(x_i-1)(x_t-1)\\
&=\sum_{i=1}^{t} p_i(k-1)(x_i-1)(x_t-1)+\sum_{i=t+1}^{r} p_i(k-1)(x_t-1)(x_i-1)\\
&\quad+\sum_{i=t+1}^{r} p_i(k-1)([x_i,x_t]-1)
\end{aligned}
$$

Hence

$$y\sigma_2 = \sum_{i=1}^{t} p_i\bar{k}\otimes\bar{x}_i\vee\bar{x}_t + \sum p_i\bar{k}\otimes(\bar{x}_t\vee\bar{x}_i) + \sum p_i\bar{k}\otimes\overline{[x_i,x_t]} + R_3(H,K).$$

$$= \bar{k}\otimes\bar{a}_0\vee\bar{h}_t + \bar{k}\otimes\overline{[a_0,h]}$$

$$= \bar{k}\otimes\bar{a}\vee\bar{h}_t + \bar{k}\otimes\overline{[a_0,h]}$$

$$\in ((K_1/K_2)\otimes([H,K]\vee H))\oplus((K_1/K_2)\otimes(H_2/H_{(3)})).$$

By Lemma 7, σ is an isomorphism θ being the inverse. σ induces a homomorphism σ' on

$$\frac{A_K A_{[H,K]} A_H + A_K^3 A_H + A_K^2 A_H^2 + A_K A_H^3}{A_K^3 A_H + A_K^2 A_H^2 + A_K A_H^3} = X \quad \text{say}.$$

Define $\psi : ((K_1/K_2)\otimes([H,K]\vee H))\oplus((K_1/K_2)\oplus(H_2/H_{(3)}))\to X$ such as follows. $\psi = \psi_1 + \psi_2$ where ψ_1 is defined on the ordered triples (k,a,h); $k\in K_1K_2$, $a\in[H,K]$, $h\in H$.

$$(k,a,h)\psi_1 = (k-1)(a-1)(h-1) + A_K^3A_H + A_K^2A_H^2 + A_KA_H^3$$

$$= (k-1)(a_0-1)(h-1) + A_K^3A_H + A_K^2A_H^2 + A_KA_H^3.$$

ψ_1 is clearly trilinear and therefore induces a homomorphism $\psi_1 : (K_1/K_2)\otimes([H,K]\vee H)\to X$. ψ_2 is defined by $(k,h)\to(k-1)(h-1)+X$; $k\in K$, $h\in H_2$, ψ_2 is also bilinear and hence induces a homomorphism $\psi_2 : (K_1/K_2)\otimes(H_2/H_{(3)})\to X$.

It is easy to verify that $\sigma'\psi$ is the identity map on X while $\psi\sigma'$ is the identify homomorphism on

$$([K_1/K_2)\otimes(H_2/H_{(3)})]\oplus[(K_1/K_2)\otimes([H,K]\vee H)].$$

Since σ,σ' are isomorphisms, $\bar{\sigma}$ is an isomorphism. Hence the theorem.

Corollary 9. *If $G = H\times K$ is finite with H and K forming direct factors, than*

$$A_G^2A_H/A_G^3A_H \simeq [(H_3/H_4)\oplus(H_1/H_2\otimes H_2/H_3)\oplus \mathrm{Sp}^3(H_1/H_2)]/R$$

$$\oplus\{[W_2(K)\otimes(H_1/H_2)]\oplus[(K_1/K_2)\otimes W_2(H)]\}/R_3(H,K)$$

where R is the submodule of $(H_3/H_4)\oplus(H_1/H_2\otimes H_2/H_3)\oplus\mathrm{Sp}^3(H_1/H_2)$ generated by elements

$$\frac{d(j)}{d(i)}\,\overline{[x_{1i}^{d(i)},x_{1j}]} + \left\{\frac{d(j)}{d(i)}\,(\bar{x}_{1j}\otimes\overline{x_{1i}^{d(i)}}) - (\bar{x}_{1i}\otimes\overline{x_{1j}^{d(j)}})\right\}$$

$$+\left\{\binom{d(j)}{2}(\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1i}) - \frac{d(j)}{d(i)}\binom{d(i)}{2}(\bar{x}_{1i}\vee\bar{x}_{1i}\vee\bar{x}_{1i})\right\}$$

with $1\le i\le j\le\lambda(1)$ and $R_3(H,K)$ is the subgroup of $[W_2(K)\otimes(H_1/H_2)]\oplus[(K_1/K_2)\otimes W_2(H)]$ generated by elements $(\bar{y}_i^m\otimes\bar{x}_j - \bar{y}_i\otimes\bar{x}_j^m)$ where $m=[d(i),d'(i)]$ is the least common multiple of $d(i)=o^*(x_i)$ and $d'(i)=o^*(y_i)$; and

$$\frac{[d(j), d'(i)]}{d'(i)} \{[\overline{y_{1i}^{d'(i)}} - \binom{d'(i)}{2}(\bar{y}_i \vee \bar{y}_i) \otimes \bar{x}_j\}$$

$$= \frac{[d(j), d'(i)]}{d(j)} \left\{\bar{y}_i \otimes \left[\overline{x_j^{d(j)}} - \binom{d(j)}{2}(\bar{x}_j \vee \bar{x}_j)\right]\right\}.$$

Proof. From Lemma 5, since H and K commute,

$$A_G^2 A_H = A_H^3 + A_K A_H^2 + A_K^2 A_H$$

and

$$A_G^3 A_H = A_H^4 + A_K A_H^3 + A_K^3 A_H + A_K^2 A_H^2.$$

Hence

$$A_G^2 A_H / A_G^3 A_H \simeq (A_H^3 / A_H^4) \oplus \left(\frac{A_K^2 A_H + A_K A_H^2}{A_K A_H^3 + A_K^2 A_H^2 + A_K^3 A_H} \right).$$

The result follows by Lemma 7 and [4, Theorem 7].

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